On the relations between quantum fields and local algebras of observables over curved space-time

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Abstract. *It is shown that quantum fields over a curved space-time with a transitive group of isometries are well-defined objects at each space-time point in the meaning of sesquilinear forms. If these quantum fields are associated with a local net of observables then they can be obtained as limits of sequences of observables measurable in space-time regions shrinking to a given point.*

1. INTRODUCTION

In the last years the relations between the general theory of quantized fields and the algebraic relativistic quantum theory have been intensively studied and some open problems have been solved (see e.g. [1] – [5] and references there). In the former the main objects are operator-valued distributions (called quantum fields) and in the latter the main object is a local net of algebras (called local algebras of observables). In particular a method of recovering quantum fields from a given local net of algebras has been presented in [1] – [4]. An interesting result proved and used in these papers is that quantum fields are well-defined objects at each space-time point in the meaning of sesquilinear forms, i.e. the expectation values of the quantum fields are continuous functions over space-time. The proofs are essentially the translation covariance, spectrality, and strong continuity of the quantum fields. The underlying space-time is the Minkowski space-

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time.

At the same time the general theory of quantized fields and the algebraic relativistic quantum theory over curved space-time has been successfully investigated in a number of recent papers (see e.g. [6] – [9] and the monograph [10] for details). Therefore it seems to be interesting to extend the cited results – about the pointwise meaning of quantum fields and about the recovering of quantum fields from local nets of algebras – to the case of curved space-time. This is the aim of the present paper. Here a main assumption is that the space-time manifolds have a transitive group of isometries. Naturally the proofs in this paper considerably differ from those in [1] – [4] because there is no spectrality and no transitive abelian group of isometries, in general. The results are nearly the same. In difference to the results for the Minkowski space-time the expectation values of the quantum fields are only local continuous functions over space-time. If the space-time manifold is Minkowskian then the results are also new in case that there is no spectrality and/or the test function space for the quantum fields is $C_0^\infty(\mathbf{R}^n)$. In the present paper all results are proved for the test functionc space $C_0^\infty(\mathcal{M})$. One can replace it by another space $\mathcal{C} \subset C_0^\infty(\mathcal{M})$ provided the structure of $\mathcal{L}$ is compatible with the action of the group of isometries.

2. QUANTUM FIELDS AND LOCAL NETS OF ALGEBRAS

In this section we introduce the necessary definitions, assumptions, and notation.

Let $\mathcal{M}$ be a connected C^∞-manifold of dimension $n \geq 1$. We assume that there is a metric G on $\mathcal{M}$, i.e. a continuous symmetric tensor field $G_{ij}(x)$ which is non-degenerate. For simplicity we also assume that the components of the metric tensor are C^∞-functions on $\mathcal{M}$. Thus $\mathcal{M}$ has (pseudo) – Riemannian structure. Further we assume that there is given a transitive connected Lie group $\mathcal{G}$ of isometries on $\{\mathcal{M}, G\}$. We say that $\mathcal{M}$ is equipped with a causality structure if there is a special map $\perp$ from the open subsets of $\mathcal{M}$ into the open subsets of $\mathcal{M}; \perp\colon \mathcal{M} \supset \mathcal{O} \to \mathcal{O}^\perp \subset \mathcal{M}, \mathcal{O}^\perp$ is the causal complement of $\mathcal{O}$, the properties of $\perp$ are not important here). Next we require that there exists a strongly continuous unitary representation $U(g)$ of the group $\mathcal{G} \ni g$ on a separable Hilbert space $\mathcal{K}$.

Now we describe what we mean by «quantum field» in this paper.

DEFINITION 1. A quadruple $\{\mathcal{K}, U(\cdot), \mathcal{D}, A(\cdot)\}$ is called a *quantum field* if it satisfies the following assumptions: (i) $A(\cdot) : C_0^\infty(\mathcal{M}) \ni\!\to A(f)$ is a linear map into the set of closable operators on $\mathcal{K}$. Further, dom $A(f) = \mathcal{D}$ and $A(f^*) \restriction \mathcal{D} = A(\bar{f})$ for all $f \in C_0^\infty(\mathcal{M})$. If $\{V, x\}$ is a chart of $\mathcal{M}$ and supp $f \subset V$, then

$$A(f) = A_x((\det G_x)^{\frac{1}{2}} f_x)$$

where f_x, G_x represent f, G in the chart $\{V, x\}$ and $A_x(\cdot)$ is a distribution on $C_0^\infty(x\mathcal{V})$.

(ii) For fixed $u \in \mathcal{D}$ the map $C_0^\infty(\mathcal{M}) \ni f \to A(f)u$ is strongly continuous.

(iii) There is a countable set $\mathcal{D}_c \subseteq \mathcal{D}$ which is a core for all $A(f)$, i.e. $\overline{A(f) \restriction \mathcal{D}_c} = \overline{A(f)}$.

(iv) $\mathcal{D}$ is invariant under $U(\cdot)$, i.e. $U(g)\mathcal{D} \subseteq \mathcal{D}$ for all $g \in \mathcal{G}$. Further, $U(g) A(f)U(g)u = A(f_g)u, u \in \mathcal{D}$, with $f_g = fg^{-1}$.

If there is a causal structure $\perp$ on $\mathcal{M}$ we call the quantum field causal if the relation

$$(A(f_1)u, A(f_2)v) = (A(\overline{f}_2)u, A(\overline{f}_1)v) \tag{1}$$

holds with $u, v \in \mathcal{D}$, supp $f_i \subset \mathcal{O}_i$, and $\mathcal{O}_1 \subset \mathcal{O}_2^\perp$.

REMARK 1. Assume we have a quantum field theory over such a (pseudo) Riemannian manifold $\{\mathcal{M}, G\}$. This is, we have a set of operator-valued linear forms $A_j(\cdot)$, $j = 1, 2, \ldots, N$, on $C_0^\infty(\mathcal{M})$ satisfying (i), (ii), (iv) of Definition 1 $A(\cdot)$ replaced by $A_j(\cdot)$) and the invariance of the domain $\mathcal{D}(A_j(f)\mathcal{D} \subseteq \mathcal{D})$ and we have a vacuum vector $\omega(U(g)\omega = \omega$ for $g \in \mathcal{G})$ which is cyclic for the polynomial algebra $\mathcal{P}(A_j)$ in the fields $A_j(f)$. Then it is easy to prove (iii) for the set $\mathcal{D} = \mathcal{P}(A_j)\omega$. This means «quantum fields» of a quantum field theory are also quantum fields in the meaning of Definition 1 (the idea of the proof can be taken form [12]). For a definition of a quantum field theory over curved space-time see e.g. [7], [9], [10].

REMARK 2. It is possible to generalize Definition 1 in the following way:

1. For each open set $\mathcal{O}$ there is a dense set $\mathcal{D}(\mathcal{O})$ in $\mathcal{K}$ with $\mathcal{D}(\mathcal{O}_1) \subseteq \mathcal{D}(\mathcal{O}_2)$ if $\mathcal{O}_1 \supseteq \mathcal{O}_2$ and $\mathcal{D} = \cup_o \mathcal{D}(\mathcal{O})$.

2. dom $A(f) = \mathcal{D}(\mathcal{O})$ with supp $f = \bar{\mathcal{O}}, \mathcal{O} = \operatorname{int} \bar{\mathcal{O}}$. (this means the domain of $A(f)$ can depend on the support of f).2. The domains $\mathcal{D}(\mathcal{O})$ are to transform covariant with respect to $\mathcal{G}$. Later we will see that it is sometimes useful and necessary to introduce such «local» domains. Now we come to the local net of algebras.

DEFINITION 2. A triple $\{\mathcal{K}, U(\cdot), \mathcal{A}\}$ is called a *local net of algebras* if it satisfies the following assumptions: (i) To each open region $\mathcal{O} \subset \mathcal{M}$ there is assigned a von Neumann algebra $\mathcal{A}(\mathcal{O})$ on $\mathcal{K}$ and $\mathcal{A} = \cup_o \mathcal{A}(\mathcal{O})$ (net).

(ii) $\mathcal{O}_1 \supseteq \mathcal{O}_2$ implies $\mathcal{A}(\mathcal{O}_1) \supseteq \mathcal{A}(\mathcal{O}_2)$ (isotony).

(iii) $U(g)\mathcal{A}(\mathcal{O})U(g)^* = \mathcal{A}(g\mathcal{O})$ for all $g \in \mathcal{G}$ and all open regions $\mathcal{O}$ (covariance).

If there is a causal structure $\perp$ on $\mathcal{M}$, then we call the local net $\{\mathcal{K}, U(\cdot), \mathcal{A}\}$ causal if

$$A_1 A_2 = A_2 A_1 \text{for all } A_i \in \mathcal{A}(\mathcal{O}_i),\ i = 1, 2,$$

with $\mathcal{O}_1 \subset \mathcal{O}_2^\perp$ (for a definition of a local net of algebras of observables over curved space-time and some assertions see e.g. [6], [8]).

A natural assumption for the connection between local nets of algebras and quantum fields (belonging to the same theory) is that the relation

$$(2) \qquad (A(\bar{f})v, Cu) = (v, CA(f)u)$$

holds with $u, v \in \mathcal{D}$, $C \in \mathcal{A}(\mathcal{O})'$, and supp $f \subset \mathcal{O}$. We say that $A(\cdot)$ is relatively local to $\{\mathcal{K}, U(\cdot), \mathcal{A}\}$ if (2) is fulfilled. From the strong continuity of the quantum field it follows that (2) is also true for f with supp $f \in \bar{\mathcal{O}}$.

PROPOSITION 1. *Let $\{\mathcal{K}, U(\cdot), \mathcal{D}, A(\cdot)\}$ be a quantum field which is relativity local to a net of algebras $\{\mathcal{K}, U(\cdot), \mathcal{A}\}$. Set $\mathcal{D}(\mathcal{O}) := \{Cv; v \in \mathcal{D},\ C \in \mathcal{A}(\mathcal{O})'\}$. Then the extension $A_{\mathcal{O}}(f) := \overline{A(\bar{f})^* \restriction \mathcal{D}(\mathcal{O})}$ of $A(f)$ is affiliated to $\mathcal{A}(\mathcal{O})(\bar{\mathcal{O}} =$ supp$f)$. In particular we have $Cu \in$ dom$A_{\mathcal{O}}(f)$ and $A_{\mathcal{O}}(f)Cu = CA_{\mathcal{O}}(f)u$ if $u \in$ dom $A_{\mathcal{O}}(f)$ and $A_{\mathcal{O}}(f)Cu = CA_{\mathcal{O}}(f)u$ if $u \in$ dom $A_{\mathcal{O}}(f)$ and $C \in \mathcal{A}(\mathcal{O})'$.*

Proof. The proof is the same as in [3, Proposition 1]. ■

We can consider the quadruple $\{\mathcal{K}, U(\cdot), \mathcal{D}(\cdot), A_{\mathcal{O}}(\cdot)\}$ as a quantum field with «local» domain (see Remark 2).

At the end of this section we note some facts about the local coordinates on $\mathcal{M}$ given by the transitive connected Lie group $\mathcal{G}$ of isometries (see e.g. Pontrjagin [11, ch. 7]). Let P be a point of $\mathcal{M}$. The isotropy group of P (given by $\mathcal{G}$) is denoted by $\mathcal{S}_p$. $\mathcal{M}$ is diffeomorph to $\mathcal{G}/\mathcal{S}_p$. Further, there is a submanifold $\mathcal{G}_p$ of $\mathcal{G}$ (no subgroup in general) and a neighbourhood $\mathcal{V}_p$ of P such that the map

$$\mathcal{G}_p \ni g \to P_g := gP \in \mathcal{V}_p$$

is one to one and for $\mathcal{G}_p$ there is one chart $\{\mathcal{G}_p, \chi_p\}$. Put $\chi_p \mathcal{G}_p = U_p \subset \mathbb{R}^n$. Thus we can parametrize $\mathcal{G}_p$ by these coordinates: $g = \chi_p^{-1}(z) =: g(z), z \in U_p, g(0) = I$ (identity in $\mathcal{G}$). Correspondingly we define a chart $\{\mathcal{V}_p, x_p\}$ for the neighbourhood $\mathcal{V}_p$ of P by

$$\mathcal{V}_p \ni P_g := gP = g(z)P \to z \in U_p$$

with $x_p P_g := z = \chi_p g$. Sometimes we write $P_g = P(z)$. Because $\mathcal{G}$ is transitive we can use this chart to define charts for arbitrary points P'. Let $g' \in \mathcal{G}$ be such that $g'P = P'$. Then we define a chart $\{V_{p'}, x_{p'}\}$ by $\mathcal{G}p' := g'\mathcal{G}_p g'^{-1}$, $\mathcal{V}_{p'} := g'\mathcal{V}_p$, and $x_{p'} := x_p g'^{-1}$.

Further we consider the following functions. Let $z, z' \in U_p$. We set $P(z,z') := g(z)^{-1}g(z')P = g(z)^{-1}P(z')$. For small $|z-z'|$ the point $P(z,z')$ is in $\mathcal{V}_p$. Then we write $x_pP(z,z') = w =: \gamma(z,z')$ for the coordinates w of the point $P(z,z')$. Correspondingly we write $\gamma_1(z,z')$ for the coordinates of the point $g(z)g(z')P \in \mathcal{V}_p$. γ, γ_1 are C^∞-functions of z, z'. From the definition of γ it follows that $\gamma(o,z') = z'$ and $\gamma(z,z) = 0$. Further we get that there exists nonempty open sets $U_{3,p} \subset U_{2,p} \subset U_{1,p} \subset U_p$ such that for all $z' \in U_{2,p}$, $w \in U_{3,p}$ the function $w = \gamma(z,z')$ is invertible with respect to $z \in U_{1,p}$. This means there is a C^∞-function$k(w,z')$ from $U_{3,p} \times U_{2,p}$ into $U_{1,p}$ with $z = k(w,z')$. The existence of the function $k(w,z')$ secures that we can parametrize the region $\mathcal{W}_p := x_p^{-1}U_{2,p}$ outgoing from the points $P(w) = P' \in \mathcal{N}_p := x_p^{-1}U_{3,p}$ with the help of $\mathcal{G}_{P(w)} := \{g(z) \in \mathcal{G}_p; z = k(w,z'),\ z' \in U_{2,p}\} \subset \mathcal{G}_p$. This fact and the properties of the functions $\gamma(z,z'), \gamma_1(z,z')$, $k(w,z'$ will be used in the following sections.

3. QUANTUM FIELDS AT SPACE-TIME POINTS

The aim of this section is to show that quantum fields are well- defined at each space-time point in the meaning of sesquilinear forms. First we extend the domain of the quatntum fields and introduce a dense set of vectors for each space-time point.

From the facts that $\mathcal{G}$ is a Lie group, $A(f)u$ is strongly continuous in f, and $A(f)$ is covariant we get that

$$A(f)U(g)^*u = U(g)^*U(g)A(f)U(g)^*u = U(g)^*A(f_g)u$$

is strongly continuous in $g \in \mathcal{G}$. Let $w \in C_0^\infty(\mathcal{G})$. Then we consider the set $\mathcal{D}_\mathcal{G}$ of vector u given by

$$u = w(\mathcal{G})\tilde{u} := \int_\mathcal{G} w(g)U(g)^*\mu(\mathrm{d}g)\tilde{u},\ \tilde{u} \in \mathcal{D},$$

where μ is the Haar measure on $\mathcal{G}$. Obviously $\mathcal{D}_\mathcal{G}$ is invariant under $U(g)$, $g \in \mathcal{G}$. From this definition and the strong continuity of $A(f)U(g)^*u$ in g it follows that $\mathcal{D}_\mathcal{G} \subset \mathrm{dom}\,\overline{A(f)}$ for all $f \in C_0^\infty(\mathcal{M})$. It is easy to show that $\mathcal{D}_\mathcal{G}$ is also a core for all $A(f)$ and it contains a countable set which is a core for all $A(f)$. Moreover we see that for fixed $u \in \mathcal{D}$ and $f \in C_0^\infty(\mathcal{M})$ the map

$$C_0^\infty(\mathcal{G}) \ni w \to A(f)w(\mathcal{G})u$$

is strongly continuous and $A(f)$ is also strongly continuous in $f \in C_0^\infty(\mathcal{M})$ on the larger domain $\mathcal{D} \cup \mathcal{D}_\mathcal{G}$. Thus we can assume in the following that $\mathcal{D}$ contains $\mathcal{D}_\mathcal{G}$.

Next we come to a domain for each point $P \in \mathcal{M}$. Let $x_p, \mathcal{W}_p, \mathcal{N}_p, \mathcal{G}_p$ and the coordinates of $\mathcal{G}_p, \mathcal{V}_p$ be as described at the end of the last section. Let $w \in C_0^\infty(U_{2,p})$,

$(U_{2,p}) = x_p \mathcal{W}_p$. Let $G_p(x)$ be the metric tensor in the chart $\{\mathcal{V}_p, x_p\}$. Then we define a bounded operator $w(\mathcal{G}_p)$ by

$$w(\mathcal{G}_p)u := \int \mathrm{d}\,x(\det G_p(x))^{1/2}\, w(x)U(g(x))^* u,\ u \in \mathcal{K},$$

where $g(x)$ are the parametrized elements of $\mathcal{G}_p$. The set of all such vectors $w(\mathcal{G}_p)u$ (if u, w run through $\mathcal{K}, C_0^\infty(U_{2,p})$ and if $\mathcal{G}_p, \mathcal{V}_p, \mathcal{W}_p$ run through all admissible candidates) is denoted by $\mathcal{K}_p$. $\mathcal{K}_p$ is dense in $\mathcal{K}$ and it is covariant, $U(g)\mathcal{K}_p = \mathcal{K}_{gP}$ for all $g \in \mathcal{G}$ (use $\mathcal{G}_{p'} = g\mathcal{G}_p g^{-1}$ for $P' = gP$ and the isometry of g). Further we see that the set $\mathcal{K}(\mathcal{N}_p) := \bigcap_{p' \in \mathcal{N}_p} \mathcal{K}_{p'}$ is dense in $\mathcal{K}$ (use the covariance of $\mathcal{K}$ and the fact that the subset $\{\gamma g(z)\gamma^{-1};\ z \in U_{2,p}\}$ with $\gamma \in \mathcal{G}_p$ and $\gamma P \in \mathcal{N}_p$ also leads to coordinates of a neighbourhood of P). We omit the simple proofs.

Now we need some technical results. Let $v \in \mathcal{D}_\mathcal{G}$. Then we have $v = f_2(\mathcal{G})u$ with $f_2 \in C_0^\infty(\mathcal{G}), u \in \mathcal{D}$. We consider the map

$$C_0^\infty(\mathcal{M}) \times C_0^\infty(\mathcal{G}) \ni f_1 \times f_2 \to J_u(f_1 \times f_2) := A(f_1)f_2(\mathcal{G})u.$$

According to our preceding remarks this map is separately strongly continuous in f_1 and f_2. If $\{\mathcal{V}, x\}$ is chart of $\mathcal{M}, \{\mathcal{U}, \chi\}$ is a chart of $\mathcal{G}$, supp $f_1 \subset \mathcal{V}$, and supp $f_2 \subset \mathcal{U}$ then we get

$$J_u(f_1 \times f_2) = J_{u,v\times u}((\det G_\mathcal{V})^{1/2}\, f_{1,\mathcal{V}} \times D_\mathcal{U} f_{2,\mathcal{U}})$$

where $G_\mathcal{V}, f_{1,\mathcal{V}}, f_{2,\mathcal{U}}$ represent G, f_1, f_2 in the corresponding charts. $D_\mathcal{U}\mathrm{d}\,y$ represents the Haar measure in the chart $\{\mathcal{U}, \chi\}$ and $J_{u,v\times u}$ is a separately strongly continuous map from $C_0^\infty(x\mathcal{V}) \times C_0(\chi\mathcal{U})$ in $\mathcal{K}$. The following lemma extends the domain of J_u.

LEMMA 1. *Let $\Delta_1 \subset \mathcal{M}$, $\Delta_2 \subset \mathcal{G}$ be open bounded sets. Then the map $J_u \restriction C_0^\infty(\bar{\Delta}_1) \times C_0^\infty(\bar{\Delta}_2)$ extends to a strongly continuous map from $C_0^\infty(\bar{\Delta}_1) \otimes C_0^\infty(\bar{\Delta}_2) = C_0^\infty(\bar{\Delta}_1 \times \bar{\Delta}_2)$ to $\mathcal{K}$.*

Proof. It is sufficient to prove it for one chart because we can cover $\bar{\Delta}_1 \times \bar{\Delta}_2$ by finitely many charts. Thus the proof of the lemma reduces to the case where $\Delta_1 \subset \mathbb{R}^n$, $\Delta_2 \subset \mathbb{R}^m$ are open bounded regions. The strong continuity of $J_u(f_1 \times f_2)$ implies the continuity of the map

$$C_0^\infty(\bar{\Delta}_1) \times C_0^\infty(\bar{\Delta}_2) \ni f_1 \times f_2 \to (w, J_u(f_1 \times f_2))$$

for each vector $w \in \mathcal{K}$. The kernel theorem says that this map can be continuously extended to a map from $C_0^\infty(\bar\Delta_1 \times \bar\Delta_2)$ in $\mathbb{C}$. We denote this map by $W_{w,u}(f) = (w, J_u(f))$. The weak continuity of $J_u(f)$ implies the strong by the following argument: Consider the map

$$C_0^\infty(\bar\Delta_1 \times \bar\Delta_2) \times C_0^\infty(\bar\Delta_1 \times \bar\Delta_2) \ni g_1 \times g_2 \to$$
$$\to (J_u(g_1, J_u(g_2))$$

which is separately continuous in g_1 and g_2. Again by the kernel theorem it extends to a map which is jointly continuous. This implis that the map

$$C_0^\infty(\bar\Delta_1 \times \bar\Delta_2) \ni g \to (J_u(g), J_u(g)) = ||J_u(g)||^2$$

is continuous. Therefore $J_u(f)$ is strongly continuos in $f \in C_0^\infty(\bar\Delta_1 \times \bar\Delta_2)$. ∎

Let $P \in \mathcal{M}$ and let $\mathcal{G}_p, \mathcal{V}_p, \mathcal{W}_p, U_{2,p} = x_p\mathcal{W}_p, \gamma(z, z')$ be as described at the end of the last section. Let $h \in C_0^\infty(U_{2,p})$. Further let $\delta_{n,p}$ be an approximate delta function for the point P. That is, in the chart $\{\mathcal{V}_p, x_p\}$ it is represented by a sequence of functions $\delta_n \in C_0^\infty(U_{2,p})$ with $\delta_n(z) \geq 0$, $\left|\int \mathrm{d}z(\det\, G_p(z))^{1/2}\delta_n(z)f(z) - f(0)\right| \to 0$ as $n \to \infty$, and the support of $\delta_n(z)$ shrinks to the point zero. We set $(\det G_p(z))^{1/2}\delta_n\ (z) =: \zeta_n(z)$. Let $f_2 \in C_0^\infty(\mathcal{G})$. For simplicity suppose that the support of $f_z := f_2 \circ g(z)^{-1}, z \in U_{2,p}$, is in one chart $\{U, \kappa\}$ of $\mathcal{G}$. Let $\gamma_2(z, x_2) = x_3$ be the coordinates of the point $g(z)^{-1}g_1(x_2) \in U \subset \mathcal{G}$ where $g_1(x_2)$ is the parametrization of the elements $g_1 \in U$ given by the chart $\{U, \kappa\}$. Further, let f_2 also denote the representation of f_2 in the chart $\{U, \kappa\}$. Set $\Delta = \kappa U \subset \mathbf{R}^m$ and

$$g_n(x_1, x_2) := \int \mathrm{d}z f(\det\ G_p(z))^{1/2}h(z)$$
$$\delta_n(\gamma(z, x_1)) f_2(\gamma_2(z,\ x_2))$$

LEMMA 2. *The functions $g_n(x_1,\ x_2)$ have for all sufficiently large n support in $U_{2,p} \times \Delta$ and tend in $C_0^\infty(U_{2,p} \times \Delta)$ topology to a function $g_\infty(x_1, x_2)$.*

Proof. Since the support of $\zeta_n(w)$ tends to the point zero, the support of the functions $\zeta_n(\gamma(z, x_1))$ in x_1 for fixed $z \in \operatorname{supp} h \subset U_{2,p}$ shrinks to the point $x_1 = z$. This and the fact that $f_2(\gamma_2(z, x_2))$ has support in Δ for the z-values in question imply that $g_n(\cdot, \cdot) \in C_0^\infty(U_{2,p} \times \Delta)$ for all sufficiently large n. Further we have the existence of the C^∞-function$k(w, x_1) = z$ for $x_1 \in U_{2,p}$ and $w \in U_{3,p}$. For sufficiently large n

the values of x_1 and w in the integrand defining $g_n(x_1, x_2)$ satisfy this assumption. This gives

$$
\begin{aligned}
g_n(x_1, x_2) &= \int \mathrm{d}\, z[(\det\ G_p(z))^{1/2} h(z)\delta_n(\gamma(z, x_1)) \\
&\quad f_2(\gamma_2(z, x_2))] \\
&= \int \mathrm{d}\, w[(\det\ G_p(k(w, x_1)))^{1/2} h(k(w, x_1))\delta_n(w) \times \\
&\quad \times f_2(\gamma_2(k(w, x_1), x_2))\mathrm{D}(w, x_1)] = \\
&= \int \mathrm{d}\, w \zeta_n(w) F(x_1, x_2, w)
\end{aligned}
$$

where $\mathrm{D}(w, x_1)$ is the Jacobian determinant. The function $F(x_1, x_2, w)$ is a C^∞–function in x_1, x_2 and a C^∞-function in x_1, x_2, w. This implies that $g_n(\cdot, \cdot)$ tends in C^∞-topology to the function $g_\infty(\cdot, \cdot) = F(\cdot, \cdot, 0)$. ■

Now we are in a position to prove the main theorem of this section.

THEOREM 1. *Let $\{\mathcal{K}, U(\cdot), \mathcal{D}, A(\cdot)\}$ be a quantum field and let $\delta_{n,p}(\cdot)$ be an approximate delta function for the point P. Then:*

(i) *The limit $A(P)[u, v] := \lim(u, A(\delta_{n,p})v)$ exists for all $u \in \mathcal{K}_p$ and all $v \in \mathcal{D}_\mathcal{G}$.*

(ii) *There is a neighbourhood $\mathcal{N}_p$ of P and a dense set $\mathcal{K}(\mathcal{N}_p) := \cap_{p' \in \mathcal{N}_p} \mathcal{K}_{p'}$ in $\mathcal{K}$ such that for all $u \in \mathcal{K}(\mathcal{N}_p)$, $v \in \mathcal{D}_\mathcal{G}$ the map $\mathcal{N}_p \ni P' \to A(P')[u, v]$ is a C^∞-function.*

(iii) *The sesquilinear form $A(P)[\cdot, \cdot]$ is covariant, i.e. $A(gP)[u, v] = A(P)[U(g)^* u, U(g)^* v]$ for $u \in \mathcal{K}_{gP}$, $v \in \mathcal{D}_\mathcal{G}$.*

(iv) *Let $f \in C_0^\infty(\mathcal{M})$ with supp $f \subset \mathcal{N}_p$ (see (ii)). Set $A(f)[u, v] := \int \mathrm{d}\, x A_p(x)[u, v] f_p(x)(\det G_p(x))^{1/2}$ where $A_p(\cdot)[u, v], f_p, G_p$ denote the representation of $A(\cdot)[u, v], f, G$ in the chart $\{\mathcal{V}_p, x_p\}$. Then $(u, A(f)v) = A(f)[u, v]$.*

Proof. (i) Let $v = f_2(\mathcal{G})\tilde{v}$ and $u = w(\mathcal{G}_p)\tilde{u}$. Then

$$
\begin{aligned}
(u, A(\delta_{n,p})v) &= (\tilde{u}, w(\mathcal{G}_p^* A(\delta_{n,p} f_2(\mathcal{G})\tilde{v}) \\
&= (\tilde{u}, w(\mathcal{G}_p^* J_{\tilde{v}}(\delta_{n,p} \times f_2)).
\end{aligned}
$$

From the covariance of the quantum field and the strong continuity of J_v (see Lemma

1) we obtain

$$
\begin{aligned}
w(\mathcal{G}_p)^* J_{\tilde{v}}(\delta_{n,p} \times f_2) &= \int \mathrm{d}z(\det G_p(z))^{1/2} \overline{w(z)} U(g(z)) \times \\
&\quad \times J_{\tilde{v}}(\delta_{n,p} \times f_2) = \\
&= \int \mathrm{d}z(\det G_p(z))^{1/2} \overline{w(z)} J_{\tilde{v}}(\delta_{n,p} \\
&\quad \circ g(z)^{-1} \times f_2 \circ g(z)^{-1} \\
&= J_v(\int \mathrm{d}z(\det G_p(z))^{1/2} \overline{w(z)} \delta_{n,p} \\
&\quad \circ g(z)^{-1} \times f_2 \circ g(z)^{-1}
\end{aligned} \tag{3}
$$

For simplicity we assume that $f_2 \circ g(z)^{-1}$ has support in one chart $\{U, \kappa\}$ for all $z \in \operatorname{supp} w(\cdot)$ (in general we have to decompose the integral into N_1-parts and f_2 into N_2-parts. But the argumentation for each of these finitely many parts is the same as in the following). With respect to the chart $\{\mathcal{V}_p \times U, x_p \times \kappa\}$ the expression (3) can be written

$$
\begin{aligned}
&J_{\mathcal{V}\times U}\Big(\Big(\int \mathrm{d}z(\det G_p(z))^{1/2} \overline{w(z)} \\
&\qquad \tilde{\zeta}_n(\gamma(z,\cdot)) \times f_{2,n}(\gamma_2(z,\cdot) D_u(\cdot)\Big)
\end{aligned} \tag{4}
$$

where $\tilde{\zeta}_n(\gamma(z, x_1)) = \delta_n(\gamma(z, x_1))\ (\det G_p(x_1))^{1/2}$. Applying now Lemma 2 we get that the integrand in this expression tends in $C_0^\infty(U_{2,p} \times \kappa U)$ – topology to a function $\tilde{g}_\infty(x_1, x_2)$. Since $J_{\tilde{v}}(\cdot)$ is strongly continuous we find $w(\mathcal{G}_p)^* J_{\tilde{v}}(\delta_{n,p} \times f_2)$ converges strongly to a vector $v(P)$ from $\mathcal{K}$. Setting $A(P)[u, v] := (\tilde{u}, v(P))$ we obtain the desired result.

(ii) That $\mathcal{K}(\mathcal{N}_p)$ is dense in $\mathcal{K}$ was already remarked ($\mathcal{N}_p$ was defined at the end of Section 2). Now we define approximate delta functions $\delta_{n,P(y)}$ for the points $P(y) \in \mathcal{N}_p$ by their representations $\delta_n(x_1 - y) := \delta_n(x_1 - y)(\det G_p(0))^{1/2}(\det G_p(y))^{1/2}$ in the chart $\{\mathcal{V}_p, x_p\}$. Replace in (3) $\delta_{n,p}$ by $\delta_{n,P(y)}$. Then the integrand in (4) has the form

$$
(\det G_p(z))^{1/2} \overline{w(z)} \tilde{\delta}_n(\gamma(z, x_1 - y)) f_{2,n}(\gamma_2(z, x_2) D_u(\cdot)).
$$

Because of Lemma 2 (replacing x_1 by $x_1 - y$) we get that the integral tends to $\tilde{g}_\infty(x_1 - y, x_2)$ in $C_0^\infty(U_{2,p} \times \Delta)$-topology and is a C^∞-function in y. The strong continuity of J_v gives then the desired result.

(iv) We work in the chart $\{\mathcal{V}_p, x_p\}$ and set $\zeta_{n,y}(x) := \zeta_n(x-y)$. Now we consider the expression

$$\int \mathrm{d}\,y(\det G_p(y))^{1/2} f(y) A_p(\zeta_{n,y}) u.$$

It is well-defined because of the strong continuity of $A_p(\cdot)$. Again the strong continuity allows to write

$$\begin{aligned}(5)\qquad &\int \mathrm{d}\,y(\det G_p(y))^{1/2} f(y) A_p(\zeta_{n,y}) u = \\ &= A_p\left(\int \mathrm{d}\,y(\det G_p(y))^{1/2} \zeta_{n,y} f(y)\right) u.\end{aligned}$$

$\zeta_{n,y}(x)$ is an approximate delta function for the point x with respect to the variable y. Thus the integral in (5) tends in C^∞-topology to the function $(\det G_p(x))^{1/2} f(x)$. This implies that (5) strongly converges to the vector $A_p(f)u = A(f)u$, where f also denotes the corresponding function on $\mathcal{M}$. On the other hand we have

$$\begin{aligned}&\left(v, \int \mathrm{d}\,y(\det G_p(y))^{1/2} f(y) A_p(\zeta_{n,y}) u\right) = \\ &= \int \mathrm{d}\,y(\det G_p(y))^{1/2} f(y)(v, A_p(\zeta_{n,y}) u).\end{aligned}$$

Because of (i) and (ii) the integrand on the right hand side tends pointwise to the C^∞-function$(\det G_p(y))^{1/2} f(y) A_p(y)[u, v]$ on $U_{3,p} \ni y$. Moreover from the proof of (ii) we see that the integrand tends uniformy on $U_{3,p}$ to the limit. Therefore we can apply the Lebesgue dominated convergence theorem and get:

$$\begin{aligned}A(f)[v,u] :&= \int \mathrm{d}\,y(\det G_p(y))^{1/2} f(y) A_p(y)[v,u] \\ &= \lim_{n\to\infty}\left(v, A_p \int \mathrm{d}\,y(\det G_p(y)\right)^{1/2} \\ &f(y) A_p(\zeta_{n,y}) u) = \\ &= (v, A(f)u), \quad v \in \mathcal{K}(\mathcal{N}_p), u \in \mathcal{D}_G.\end{aligned}$$

(iii) It follows from the covariance of the quantum field, from (i), and from the fact that $\delta_{n,p} \circ g^{-1}$ is an approximate delta function for the point gP. ∎

REMARK 3. This theorem, in particular (iv), says us that quantum fields have a meaning as sesquilinear forms at each space-time point P. The dependence on P is locally continuous. In general we have no common domain for all points $P \in \mathcal{M}$ in difference to the case where $\mathcal{M}$ is in the Minkowski space-time.

REMARK 4. The proofs have been given only in one chart. But using the definition of the quantum fields as operator-valued linear forms on $\mathcal{M}$ it is easy to check that the proofs and definition do not depend on the choice of the charts.

4. QUANTUM FIELDS AS LIMITS OF LOCAL OBSERVABLES

Assume we have a set of quantum fields $\{\mathcal{K}, U(\cdot), \mathcal{D}_j, A_j(\cdot)\}$ which are relatively local to a given local net of algebras of observables $\{\mathcal{K}, U(\cdot), \mathcal{A}\}$. Then the question is whether we can recover these quantum fields from the algebra $\mathcal{A}$. In case that $\mathcal{M}$ is the Minkowski space-time this was solved by Fredenhagen, Hertel [1] under the additional assumption that the quantum fields are energetic bounded and by Rehberg, Wollenberg [2], Wollenberg [3] for arbitrary quantum fields over the Minkowski space-time.

For the aim to recover the quantum fields from $\mathcal{A}$ it is necessary to introduce operator functions $A(\cdot)$ over $\mathcal{M}$ for elements $A \in \mathcal{A}$. Fix a point $P \in \mathcal{M}$. Choose a neighbourhood $\mathcal{V}_p$ of P, a submanifold $\mathcal{G}_p$ from $\mathcal{G}$, and a chart $\mathcal{V}_p, x_p$ as described at the end of Section 2. Take a sequence of elements $g_n \in \mathcal{G}$ such that $Ug_n\mathcal{W}_p = \mathcal{M}$. Thus the set $\{\mathcal{W}_n, x_n\}$ is an atlas for $\mathcal{M}$ where $\mathcal{W}_n := g_n\mathcal{W}_p, x_n = x_p \circ g_n^{-1}, \mathcal{W}_0 = \mathcal{W}_p$, and $g_0 = I$. Let $P_n \in \mathcal{W}_n$. Then $P_n = g_n g(z)P$ for some $z \in U_{2,p} = x_p\mathcal{W}_p$. Thus we define (in the region $\mathcal{W}_n$)

$$A(P_n) := U(g_n)U(g(z))AU(g(z))^*U(g_n)^*. \tag{6}$$

Now let $f \in C_0^\infty(\mathcal{M})$. There is a partition of the unity by functions χ_n corresponding to $\mathcal{W}_n$. That is, $\operatorname{supp}\chi_n \subset \mathcal{W}_n, \sum_n \chi_n(\tilde{P}) = 1$ for all $\tilde{P} \in \mathcal{M}$, and $\chi_n \in C_0^\infty(\mathcal{M})$. We set $f_n := f\circ\chi_n$ and denote by $\tilde{f}_n$ the corresponding function in the chart $\{\mathcal{W}_n, x_n\}$ given by $\tilde{f}_n := f_n \circ x_n^{-1}$. We set

$$\begin{aligned} A[f] &:= \textstyle\sum_n A[f_n], \\ A[f_n] &:= \int \mathrm{d}z(\det G_n(z))^{1/2}U(g_n)U(g(z))AU(g(z))^*U(g_n)^*\tilde{f}_n(z) \\ &= U(g_n)\left[\int \mathrm{d}z(\det G_0(z))^{1/2}U(g(z))AU(g(z))^*\tilde{f}_n(z)\right]U(g_n)^*. \end{aligned} \tag{7}$$

Here we have used that $\det G_n(z) = \det G_0(z) = \det G_p(z)$, because of our special choice of coordinates, and the fact that g_n are isometries. Thus the integration reduces to integration over the region $U_{2,p} \subset \mathbf{R}^n$. Naturally this definition depends on the chosen atlas. If A is invariant under the isotropy group S_p of the point P, i.e. $U(g)A = AU(g)$ for $g \in S_p$, then the definition is independent of the special atlas. Now we are able to formulate the following statement.

THEOREM 2. Let $\{\mathcal{K}, U(\cdot), \mathcal{D}, A(\cdot)\}$ be a quantum field which is relativity local to a given local net of algebras $\{\mathcal{K}, U(\cdot), \mathcal{A}\}$. Fix a point $P \in \mathcal{M}$ and choose an atlas

$\{\mathcal{W}_n, x_n\}$ as above. Then there is a sequence $A_n \in \mathcal{A}(\mathcal{O}_n)$ where $\mathcal{O}_n$ is a sequence of open bounded regions shrinking to the point P such that

$$s - \lim_{n\to\infty} \quad A_n[f]u = A(f)u, \; u \in \mathcal{D}_2,$$

for all functions $f \in C_0^\infty(\mathcal{M})$. Here $\mathcal{D}_2 \subset \mathcal{D}$ is a countable dense set which is a core for all $A(f)$ (naturally A_n does not depend on f and u).

Proof 1. First we define the set $\mathcal{D}_2$ according to the chosen atlas. We set $\mathcal{D}_2 := \{U(g_{j_1})\dots U(g_{j_m})v; v \in \mathcal{D}_0, j_s \in \mathbb{N}, m \in \mathbb{N}\}$ where $\mathcal{D}_0 \subset \mathcal{D}$ is countable core for all $A(f)$. Thus $\mathcal{D}_2 \subset \mathcal{D}$. From (7) and the invariance of $\mathcal{D}_2$ under $U(g_j)$ it follows that is sufficient to prove the theorem for all functions $f \in C_0^\infty(\mathcal{M})$ with supp $f \subset \mathcal{W}_p = \mathcal{W}_0$. Therefore we restrict ourselves to this region and use the corresponding definitions and notations introduced at the end of Section 2.

2. Let $\delta_{n,p}$ be an approximate delta function for the point P and let $f \in\in C_0^\infty(\mathcal{M})$ with supp $f \in \mathcal{W}_p$. Let $u \in \mathcal{D}$. Then

$$\begin{aligned} A(f \square \delta_{n,p})u := &\int \mathrm{d}\, z[(\det G_0(z))^{1/2} f(z) U(g(z)) \times \\ &\times A(\delta_{n,p}) U(g(z))^* u] = \\ &= \int \mathrm{d}\, z(\det G_0(z))^{1/2} f(z) A(\delta_{n,p} \circ g(z)^{-1}) u \\ &= A\left(\int \mathrm{d}\, z(\det G_0(z))^{1/2} f(z) \delta_{n,p} \circ g(z)^{-1}\right) u \\ &= A_p\left(\int \mathrm{d}\, z(\det G_0(z))^{1/2} f(z) \tilde{\zeta}(\gamma(z,\cdot))\right) u \end{aligned}$$

where we have used the strong continuity of $A(\delta_{n,p} \circ g(z)^{-1})$ in z. The integral on the right hand side tends in $C_0^\infty(\mathcal{M})$-topology to a limit. This can be proved in the same way as Lemma 2. Thus $A(f \square \delta_{n,p})u$ strongly converges to a vector $u(f)$.

3. We show that $u(f) = A(f)u$. Assume that we have proved this for $u \in \mathcal{D}_\mathcal{G}$ and functions f with supp $f \subset \mathcal{N}_p$. With the help of the partition of the unity for $\mathcal{W}_p$ and the fact that $\mathcal{D}_\mathcal{G}$ is core for all $A(f)$ we can easily extend this to all $u \in \mathcal{D}$ and $f \in C_0^\infty(\mathcal{W}_p)$.

Now we prove $u(f) = A(f)u$ for $u \in \mathcal{D}_\mathcal{G}$ and $f \in C_0^\infty(\mathcal{N}_p)$ with the help of Theorem 1 (ii), (iv). We have

$$\lim_{n\to\infty}(v, A(f\square\delta_{n,p}u) =$$

$$= \lim_{n\to\infty}\int \mathrm{d}\,z(\det G_0(z))^{1/2} f(z)\times$$

$$\times \int A_p(x)[U(g(z))^* v, U(g(z))^* u]$$

$$\zeta_n(x)\mathrm{d}\,x =$$

$$= \lim_{n\to\infty}\int \mathrm{d}\,z(\det G_0(z))^{1/2}$$

$$f(z)\int A_p(\gamma_2(z,x))[v,u]\zeta_n(x)\mathrm{d}\,x =$$

$$= \int \mathrm{d}\,z(\det G_0(z))^{1/2} f(z) A_p(z)[v,u] =$$

$$= A(f)[v,u] = (v, A(f)u)$$

where $\gamma_2(z,x)$ are the coordinates of the point $g(z)P(x)$.

The interchange of the integral with the limit is an easy consequence of the C^∞– properties of $\gamma_2(z,x)$ and $A_p(y)[v,u]$. This gives $u(f) = A(f)u$.

4. In Proposition 1 we introduced the operators $A_{\mathcal{O}}(f)$ which are affiliated to the algebra $\mathcal{A}(\mathcal{O})$. Thus $A_{\mathcal{O}}(f)^*$ is also affiliated to $A_{\mathcal{O}}$. Let $A_{\mathcal{O}}(\delta_{n,p})^* = U_n \; |A_{\mathcal{O}}(\delta_{n,p})^*|$ be the polar decomposition and let $P_n(\Delta)$ be the spectral projections of $|A_{\mathcal{O}}(\delta_{n,p})^*|$. Then $P_n(\Delta)A_{\mathcal{O}}(\delta_{n,p}) = P_n(\Delta)|A_{\mathcal{O}}(\delta_{n,p})^*|U_n^*$ and $(P_n(\Delta) \;\; A_{\mathcal{O}}(\delta_{n,p}))^*$ are bounded operators belonging to $\mathcal{A}(\mathcal{O})$ if δ is a bounded interval of $\mathbf{R}$. Set $W_n(\Delta) := U_n P_n(\Delta) \;\; U_n^*$. Then we get $(P_n(\Delta)A_{\mathcal{O}}(\delta_{n,p}))^* = W_n(\Delta)A_{\mathcal{O}}(\delta_{n,p}))^*$. Now we define

$$A_m := (1/2)[P_m(\Delta_m)A_{\mathcal{O}}(\delta_{m,p}) + W_m(\Delta_m)A_{\mathcal{O}}(\delta_{m,p})^*] \tag{8}$$

By definition we have $A_m^* = A_m$ and $A_m \in \mathcal{A}(\mathcal{O}_m)$ where $\mathcal{O}_m$ is a sequence shrinking to the point P because of the properties of the support of $\delta_{m,p}$. Next we want to prove that

$$(A_m[f] - A(f))u$$

tends strongly to zero for $u \in \mathcal{D}_2, f \in C_0^\infty(\mathcal{W}_p)$. Inserting the definition of $A_m[f]$

and using $A_{\mathcal{O}}(f)u = A(f)u = A(f)^*u$ for $u \in \mathcal{D}, f = \bar{f}$ we obtain

$$(9)\quad \begin{aligned} \| A_m[f] - A(f)u \| &\leq (1/2)(\| \int \mathrm{d}\, x(\det G_0(z))^{1/2} f(z) \times \\ &\times U(g(z)) P_m(\Delta_m) U(g(z))^* A(\delta_{m,p} \circ g(z)^{-1})u - A(f)u \| + \\ &+ \| \int \mathrm{d}\, z (\det G_0(z))^{1/2} f(z) U(g(z)) W_m(\Delta_m) U(g(z))^* \times \\ &\times A(\delta_{m,p} \circ g(z)^{-1})u - A(f)u \|) \leq \\ &\leq \| A(f \square \delta_{m,p})u - A(f)u \| + \\ &+ (1/2) \int \mathrm{d}\, z (\det G_0(z))^{1/2} |f(z)| \times \\ &\times \{\| (P_m(\Delta_m) - 1) U(g(z))^* A(\delta_{m,p} \circ g(z)^{-1})u \| + \\ &+ \| (W_m(\Delta_m) - 1) U(g(z))^* A(\delta_{m,p} \circ g(z)^{-1})u \|\}. \end{aligned}$$

The first term on the right hand side tends to zero as $m \to \infty$ because of 1. To prove that the second and third term tends to zero as $m \to \infty$ it sufficiens to show that the expression

$$\begin{aligned} c(m,u) := \sup_{z \in \mathcal{U}_{z,p}} \{ \| & (P_m(\Delta_m) - 1) \\ & U(g(z))^* A(\delta_{m,p} \circ g(z)^{-1})u \| + \\ & + \| (W_m(\Delta_m) - 1) U(g(z))^* A(\delta_{m,p} \circ g(z)^{-1})u \| \} \end{aligned}$$

goes to zero as $m \to \infty$ for a suitable choice of the bounded intervalls Δ_m. Since $A(\delta_{m,p} \circ g(z)^{-1})u$ and $U(g(z))$ are strongly continuous, the set of vectors $\{U(g(z))^* A(\delta_{m,p} \circ g(z)^{-1})u; z \in \mathcal{U}_{z,p}\}$ is for fixed m and u a pre-compact set in norm. On the other hand $(P_m(\Delta) - 1)$ and $(W_m(\Delta) - 1)$ tend on compact (also on pre-compact) sets of vectors uniformly to zero as $\Delta \to \mathbb{R}$ (for fixed m). Thus, for given $\epsilon > 0, m$, and u, we obtain that $c(m,u) < \epsilon$ if we choose Δ_m large enough. If we use this fact the rest of the proof is simple. First step: Set $m = 1$ and $u = u_1 \in \mathcal{D}_2$. Choose Δ_1 so large that $c(1,u_1) < \epsilon$. Second step: Consider $c(2,u_1)$ and $c(2,u_2)$. Choose Δ_2 so large that both terms are smaller than $(\epsilon/2)$. Continuing this procedure we get that $c(m,u_s) \to 0$ as $m \to \infty$ for each $u_s \in \mathcal{D}_2$. This concludes the proof. ■

REMARK 5. This theorem is similar to the corresponding one for Minkoski space-time. One can also prove that the sequence A_m tends to $A(P)[\cdot,\cdot]$, i.e. $\lim_{m\to\infty}(u, A_m v) = A(P)[u,v]$. This means the set of limits of sequences of observables measurable in space-time regions shrinking to a point P contains all quantum fields $A(P)[\cdot,\cdot]$ which are relatively local to the given net of observables.

5. LOCAL NETS OF ALGEBRAS AND THEIR LIMITS

In the last section we have seen that special sequences $A_m \in \mathcal{A}(\mathcal{O}_m)$ belonging to regions $\mathcal{O}_m$ shrinking to a point P give us back the quantum fields. Now the question is if we can define a topology on $\mathcal{A}$ such that the limits of $\mathcal{A}$ belonging to space-time points define us always quantum fields. This question will be answered under the additional assumption that the isotropy group $\mathcal{S}_p$ of a point P (thus for all points) is compact. For instance the following examples satisfy this assumption: 1. $\mathcal{M}$-Minkowski space-time, $\mathcal{G}$-translation group $\mathbf{R}^n$. 2. $\mathcal{M} = \mathbf{R} \times S^3$-Einstein cosmos, $\mathcal{G} = \mathbf{R} = SO(4)$. 3. $\mathcal{M}$-stationary part of the de Sitter space-time, $\mathcal{G}$-translation subgroup of the de Sitter group.

It is also possible to omit the assumption that $\mathcal{S}_p$ is compact. But then the topology on $\mathcal{A}$ is more complicated and looks a little artificial.

We use here the sequence $g_n \in \mathcal{G}$, the atlas $\{\mathcal{W}_n, x_n\}$, and the definition of $A[f]$ (see (7)). Let $\tilde{\mathcal{D}}$ be a countable dense set of vectors from $\mathcal{K}$ which is invariant under the unitary operators $U(g_n)$ (like the set $\mathcal{D}_2$ in Theorem 2). We define for each vector $u \in \tilde{\mathcal{D}}, \mathcal{O} \subset \mathcal{M}$ ($\mathcal{O}$ open and bounded), and $C_0^\infty(\mathcal{O})$-norm $|\cdot|_\alpha$ a seminorm on $\mathcal{A}$:

$$p(u, \mathcal{O}, \alpha; A) := \sup_{f \in C_0^\infty(\mathcal{O})} |f|_\alpha^{-1} \, \| A[f]u \|, \; A \in \mathcal{A}.$$

A set of seminorms $\{p(u, \mathcal{O}, \alpha(u, \mathcal{O}); \cdot), u \in \tilde{\mathcal{D}}, \mathcal{O} \subset \mathcal{M}\}$ defines us a locally convex topology $\tau(\tilde{\mathcal{D}})$ on $\mathcal{A}$.

DEFINITION 3. Let $A_n = A_n^* \in \mathcal{A}(\mathcal{O}_n)$ where the regions $\mathcal{O}_n$ shrink to the point P as $n \to \infty$. Suppose that:

(i) $U(g)A_n = A_n U(g)$ for $g \in \mathcal{S}_p, n \in \mathbf{N}$.

(ii) A_n is a Cauchy sequence for a topology $\tau(\tilde{\mathcal{D}})$, i.e. $p(u, \mathcal{O}, \alpha(u, \mathcal{O}); A_n - A_m) \to 0$ as $m, n \to \infty$ for each $u \in \tilde{\mathcal{D}}, \mathcal{O} \subset \mathcal{M}$ and $\alpha(u, \mathcal{O})$.

Then we call the sequence A_n a *shrinking* $\mathcal{G}$ – *covariant* $\tau(\tilde{\mathcal{D}})$ – *sequence* for the point P.

As already remarked the invariance of A_n under the isotropy group $\mathcal{S}_p$ implies that the definition of $A_n[f]$ is independent of the special choice of the atlas (the function $A_n(P') := U(g)A_nU(g)$ on $\mathcal{M}$ with $P' = gP$ is covariant). The next theorem connects such shrinking $\mathcal{G}$-covariant $\tau(\tilde{\mathcal{D}})$-sequences with quantum fields which are relatively local to the given net $\{\mathcal{K}, U(\cdot), \mathcal{A}\}$.

THEOREM 3. *Let $\{\mathcal{K}, U(\cdot), \mathcal{A}\}$ be a local net of algebras. Let A_n be a shrinking $\mathcal{G}$-covariant $\tau(\tilde{\mathcal{D}})$-sequence for the point P. Then this sequence defines a quantum field $\{\mathcal{K}, U(\cdot), \mathcal{D}, A(\cdot)\}$, where $\mathcal{D} := \{U(g)u; u \in \tilde{\mathcal{D}}, g \in \mathcal{G}\}$, by the relation*

$$A(f)u := s - \lim_{n \to \infty} A_n[f]u$$

for $u \in \mathcal{D}$ and $f \in C_0^\infty(\mathcal{M})$. Further the quantum field is relatively local to $\{\mathcal{K}, U(\cdot), \mathcal{D}, \mathcal{A}\}$. If $\mathcal{A}$ is a causal then the quantum field is also causal.

Proof. 1. Since $A_n[f]u$ is a Cauchy sequence in $\mathcal{K}$ there is a limit vector $u(f) := s - \lim_{n\to\infty} A_n[f]$. We define an operator $A(f)$ by $A(f)u := u(f)$ on $\tilde{\mathcal{D}}$ for fixed f. Further $A(f)^*u = A(\bar{f})u$ because of the same relation for $A_n[f](A_n = A_n^*)$. Thus $A(f)$ is closable. Since $A_n[f]$ is covariant we get

$$U(g)A_n[f] = A_n[f\cdot g^{-1}]U(g), \quad g \in \mathcal{G}.$$

This shows that the sequence $A_n[f]$ also converges on the larger set $\mathcal{D} \supset \tilde{\mathcal{D}}$. That is $A(f)$ is defined on $\mathcal{D}$ and $A(f)^*u = A(\bar{f})u$ for $u \in \mathcal{D}$. The covariance of $A(f)$ easily follows from the covariance of $A_n[f]$. Thus (i) and (iv) of Definition 1. are fulfilled.

2. Now we prove (ii) of Definition 1. From Definition 3 we get

$$\begin{aligned} \| A(f)u \| \le \| \, (A(f) - A_n[f]u \| &+ \\ + \| A_n[f]u \| &\le \epsilon_n(f,u) + \\ &+ p(\mathcal{O}, u, \alpha(\mathcal{O}, u); A_n)|f|_\alpha . \end{aligned}$$

The first term on the right hand side tends to zero as $n \to \infty$. The second term can be estimated by $c(\mathcal{O},u)|f|_\alpha$ because A_n is a Cauchy sequence and therefore uniformly bounded in the seminorms. Thus we get $\| A(f)u \| \le \| \; c(\mathcal{O},u)|f|_\alpha$. This proves the strong continuity of the map $C_0^\infty(\mathcal{M}) \ni f \to A(f)u$.

3. $\mathcal{G}$ is separable. Let $\mathcal{G}_c$ be a countable dense set of $\mathcal{G}$. Then $U(\mathcal{G}_c)$ is strongly dense in $U(\mathcal{G})$. Using this, the strong continuity of $U(\cdot)$ and $A(\cdot)$, and the covariance of $A(\cdot)$ we get that $\mathcal{D}_1 := U(\mathcal{G}_c)\tilde{\mathcal{D}} \subset \mathcal{D}$ is a countable dense which is a core for all $A(f)$. The relative locality of $A(f)$ to $\mathcal{A}(\mathcal{O})$ follows by a limit procedure from the corresponding property for $A_n[f], \operatorname{supp} f \subset \mathcal{O}$. The causality of $A(\cdot)$ is a consequence of $(A_n[f]u, A_n[g]v) = (A_n[\bar{g}]u, A_n[\bar{f}]v)$. ∎

It remains to show that such shrinking sequences from $\mathcal{A}$ lead to all quantum fields related to the given algebra $\mathcal{A}$. In Section 4 we have shown that quantum fields relatively local to the algebra $\mathcal{A}$ are strong limits of $A_n[f]$. That this sequence A_n is a Cauchy sequence in some $\tau(\tilde{\mathcal{D}})$-topology follows easily from the proof of Theorem 2. The only difference is that the sequence A_n in Theorem 2 is in general not invariant under the isotropy group $\mathcal{S}_p$ of P. But this can be reached in the following way:

Replace the sequence A_m from (8) by

$$\tilde{A}_m := \left(\int_{\mathcal{S}_p} \mu(\mathrm{d}y)\right)^{-1} \int_{\mathcal{S}_p} \mu(\mathrm{d}\gamma)U(\gamma)A_m U(\gamma)^*,$$

where $\mu(d\gamma)$ is the Haar measure for the compact group S_p. Thus $\tilde{A}_m$ is now invariant under S_p. The proof of Theorem 2 works also with some small modification for this sequence $\tilde{A}_m$. One has to use that the functions $\delta_{n,p}\circ\gamma^{-1}, \gamma \in S_p$, are also approximate delta functions for the point P because of $\gamma P = P$. Thus $s-\lim_{n\to\infty} A(f\,\square\,\delta_{n,p}\circ\gamma^{-1})u = A(f)u$ for all $\gamma \in S_p$. Further use that $\{U(g(z))^* U(\gamma)^* A(\delta_{n,p}\circ\gamma^{-1}\circ g(z)^{-1})u;\ z \in U_{2,p}, \gamma \in S_p\}$ is a pre-compact set of vectors. We omit the details.

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